Chapter 2

Second-Order Differential Equations

2.1 The Linear Second-Order Equation

1. It is a routine exercise in differentiation to show that $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation, while $y_p(x)$ is a solution of the nonhomogeneous equation. The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$W(x) = \begin{vmatrix} \sin(6x) & \cos(6x) \\ 6\cos(6x) & -6\sin(6x) \end{vmatrix} = -6\sin^2(x) - 6\sin^2(x) = -6,$$

and this is nonzero for all x, so these solutions are linearly independent on the real line. The general solution of the nonhomogeneous differential equation is

$$y = c_1 \sin(6x) + c_2 \cos(6x) + \frac{1}{36}(x - 1).$$

For the initial value problem, we need

$$y(0) = c_2 - \frac{1}{36} = -5$$

so $c_2 = -179/36$. And

$$y'(0) = 2 = 6c_1 + \frac{1}{36}$$

so $c_1 = 71/216$. The unique solution of the initial value problem is

$$y(x) = \frac{71}{216}\sin(6x) - 179$$

$\cos(\frac{1}{36}(x -$

$6x)\ 1).$

$+$

2. The Wronskian of e^{4x} and e^{-4x} is

$$W(x) = \begin{vmatrix} e^{4x} & e^{-4x} \\ 4e^{4x} & -4e^{-4x} \end{vmatrix} = -8 = 0$$

so these solutions of the associated homogeneous equation are independent. With the particular solution $y_p(x)$ of the nonhomogeneous equation, this equation has general solution

$$y(x) = c_1 e^{4x} + c_2 e^{-4x} - \frac{1}{4}x^2 - \frac{1}{32}.$$

From the initial conditions we obtain

$$y(0) = c_1 + c_2 - \frac{1}{32} = 12$$

and

$$y^0(0) = 4c_1 - 4c_2 = 3.$$

Solve these to obtain $c_1 = 409/64$ and $c_2 = 361/64$ to obtain the solution

$$y(x) = \frac{409}{64}e^{4x} + \frac{361}{64}e^{-4x} - \frac{1}{4}x^2 - \frac{1}{32}.$$

3. The associated homogeneous equation has solutions e^{-2x} and e^{-x}. Their Wronskian is

$$W(x) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}$$

and this is nonzero for all x. The general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{15}{2}.$$

For the initial value problem, solve

$$y(0) = -3 = c_1 + c_2 + \frac{15}{2}$$

4. The associated homogeneous equation has solutions

$$y_1(x) = e^{3x}\cos(2x),\; y_2(x) = e^{3x}\sin(2x).$$

The Wronskian of these solutions is

$$W(x) = \begin{matrix} e^{3x}\cos(2x) & e^{3x}\sin(2x) \\ 3e^{3x}\cos(2x) - 2e^{3x}\sin(2x) & 3e^{3x}\sin(2x) + 2e^{3x}\cos(2x) \end{matrix} = e^{6x} = 0$$

for all x. The general solution of the nonhomogeneous equation is

$$y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x) - \frac{1}{8}e^{x}.$$

To satisfy the initial conditions, it is required that

$$y(0) = -1 = c_1 - \frac{1}{8}$$

and

$$3c_1 + 2c_2 - \frac{1}{8} = 1.$$

Solve these to obtain $c_1 = -7/8$ and $c_2 = 15/8$. The solution of the initial value problem is

$$y(x) = -\frac{7}{8}e^{3x}\cos(2x) + \frac{15}{8}e^{3x}\sin(2x) - \frac{1}{8}e^{x}.$$

5. The associated homogeneous equation has solutions

$$y_1(x) = e^{x}\cos(x),\ y_2(x) = e^{x}\sin(x).$$

These have Wronskian

$$W(x) = \begin{matrix} e^{x}\cos(x) & e^{x}\sin(x) \\ e^{x}\cos(x) - e^{x}\sin(x) & e^{x}\sin(x) + e^{x}\cos(x) \end{matrix} = e^{2x} = 0$$

so these solutions are independent. The general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 e^{x}\cos(x) + c_2 e^{x}\sin(x) - \frac{5}{2}e^{2} - 5x - \frac{5}{2}.$$

We need

$$y(0) = c_1 - \frac{5}{2} = 6$$

and

$$y'(0) = 1 = c_1 + c_2 - 5.$$

Solve these to get $c_1 = 17/2$ and $c_2 = -5/2$ to get the solution

$$y(x) = \frac{17}{2} e^x \cos(x) - \frac{5}{2} e^x \sin(x) - \frac{5}{2} x^2 - 5x - \frac{5}{2}.$$

6. Suppose y_1 and y_2 are solutions of the homogeneous equation (2.2). Then

$$y_1'' + py_1' + qy_1 = 0$$

and

$$y_2'' + py_2' + qy_2 = 0.$$