Aerodynamics for Engineering Students

Sixth Edition

E.L. Houghton

P.W. Carpenter

S.H. Collicott

D.T. Valentine

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E. L. Houghton, P. W. Carpenter, Steven H. Collicott

and

Daniel T. Valentine

Chapter 2 solutions

1 Solutions to Chapter 2 problems

Problem 2.1: In this problem we are interested in the continuity equation for axisymmetric flow in terms of the cylindrical coordinate system (r, ϕ, z), where all flow variables are independent of angular coordinate, ϕ. Let the velocity components $(u, v, w) = (u, 0, w)$ in the (r, ϕ, z) coordinate directions. **(a)** Show that the continuity equation is given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

Solution: Let us consider an annular control volume with square cross-sectional area equal to $drdz$ centered at (r, z). Let us label the sides of the control volume R, L, T, B the centers of which are located at $(r, z + dz/2)$, $(r, z - dz/2)$, $(r + dr/2, z)$, $(r - dr/2, z)$, respectively. In this analysis the horizontal coordinate is z and the vertical (or radial) coordinate is r. The flow through each of the surfaces is

$$\dot{m}_R = \rho \left(w + \frac{\partial w}{\partial z}\frac{dz}{2} \right) 2\pi r\, dr$$

$$\dot{m}_L = \rho \left(w - \frac{\partial w}{\partial z}\frac{dz}{2} \right) 2\pi r\, dr$$

$$\dot{m}_T = \rho \left(u + \frac{\partial u}{\partial r}\frac{dr}{2} \right) 2\pi \left(r + \frac{dr}{2} \right) dz$$

$$\dot{m}_B = \rho \left(u - \frac{\partial u}{\partial r}\frac{dr}{2} \right) 2\pi \left(r - \frac{dr}{2} \right) dz$$

The conservation of mass principle means that the net flow of mass out of the control volume is zero, i.e.,

$$\dot{m}_R - \dot{m}_L + \dot{m}_T - \dot{m}_B = 0$$

This holds for both steady and unsteady conditions because ρ is assumed to be constant. An incompressible flow is volume preserving and, hence, this result is independent of whether or not the flow is steady. This is not the case for Euler's equation of motion; see Exercise 2.4. Substituting the expressions for the mass flow across each part of the control surface and rearranging terms, we get the equation given above.

(b) Show that the Stokes's stream function ψ defined by the following expressions for the velocity components

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

By direct substitution of these expressions into the continuity equation, we see that the continuity equation is automatically satisfied. Note that the function ψ is assumed to be a function for which the order of differentiation is immaterial.

Problem 2.2: In this problem we want to transform the continuity equation given in (x, y) to polar coordinates (r, ϕ). The two coordinate systems are related by the following formulas: $x = r\cos\phi$. $y = r\sin\phi$. Let the components of the velocity in (x, y) be given by $\mathbf{u} = (u_x, u_y)$ and the components in (r, ϕ) by $\mathbf{u} = (u, v)$. Note that

$$u_x = u\cos\phi - v\sin\phi, \quad u_y = u\sin\phi + v\cos\phi$$

We want to transform

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

to the following form:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} = 0$$

What we need to recall is that the derivative of any property f transforms as follows:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$

From $x = r\cos\phi$ and $y = r\sin\phi$ we get

$$\frac{\partial x}{\partial x} = \frac{\partial r}{\partial x}\cos\phi - r\sin\phi\frac{\partial \phi}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial r}{\partial y}\sin\phi + r\cos\phi\frac{\partial \phi}{\partial y}$$

Since $r^2 = x^2 + y^2$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y$$

Hence,

$$\frac{\partial r}{\partial x} = \cos\phi, \quad \frac{\partial r}{\partial y} = \sin\phi$$

Thus,

$$\frac{\partial \phi}{\partial x} = -\frac{1-\cos^2\phi}{r\sin\phi} = -\frac{\sin\phi}{r}, \quad \frac{\partial \phi}{\partial y} = \frac{1-\sin^2\phi}{r\cos\phi} = \frac{\cos\phi}{r}$$

Applying these formulas, we get

$$\frac{\partial u_x}{\partial x} = \cos\phi\frac{\partial\left(u\cos\phi - v\sin\phi\right)}{\partial r} - \frac{\sin\phi}{r}\frac{\partial\left(u\cos\phi - v\sin\phi\right)}{\partial \phi}$$

$$\frac{\partial u_y}{\partial y} = \sin\phi\frac{\partial\left(u\sin\phi + v\cos\phi\right)}{\partial r} + \frac{\cos\phi}{r}\frac{\partial\left(u\sin\phi + v\cos\phi\right)}{\partial \phi}$$

Adding these equations and equating them to zero, we get the result sought.

Extension 1 of Problem 2.2 (see also Problem 2.7): The condition for irrotationality for two-dimensional planar flows in an (x, y) Cartesian coordinate system is:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

Transform this relationships to cylindrical polar coordinates, (r, θ), by checking and applying the following transformation relationships:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Note that:

$$u = u_r\cos\theta - u_\theta\sin\theta,$$

and

$$v = u_r\sin\theta + u_\theta\cos\theta.$$

Also note that

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}.$$

$$\frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r} \frac{\partial f}{\partial \theta}.$$

The solution is

$$\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}$$

Extension 2 of Problem 2.2 (see also Section 3.2.2): The second-derivative transformation equations are as follows:

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta \frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{2\,\cos\theta\sin\theta}{r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta} - \frac{\partial^2 f}{\partial r \partial \theta}\right).$$

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta \frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} - \frac{2\,\sin\theta\cos\theta}{r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta} - \frac{\partial^2 f}{\partial r \partial \theta}\right).$$

The second derivative relationships are useful for transforming

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

to polar coordinates. By substituting $f = \phi$ in the second-derivative relations and adding the two resulting equations, we get

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Problem 2.3: Sections 2.4 and 2.6 on continuity and momentum equations, respectively, are useful. The procedure based on control-volume analysis is applied to derive the continuity and momentum equations. The same procedure can be applied to derive the convection-diffusion equation for C in this problem.

Assume that none of the contaminant is created within the flow field. Assume that the transport of the contaminant matter is by convection and diffusion. In part (a) we assume that the velocity field is unchanged and, hence, the contaminant must be sufficiently dilute. If this is not the case, then the density of the fluid containing the contaminant could be changed in such a way as to alter the motion of the fluid in an analogous way as when temperature changes in a fluid are sufficient to cause hot air or hot water to naturally rise above cold air or cold water. This is the key concept that leads to pointing out in the problem statement that the contaminant is dilute.

The rate of increase in C in an infinitesimal control volume like the one drawn in Fig. 2.20 in the text is

$$\frac{\partial C}{\partial t}\,\delta x\,\delta y \times 1$$

The net convective transport of C across the boundary of the control volume in Fig. 2.20 is, substituting C for ρu, the horizontal component of the momentum, in Eq. (2.62a) and assuming $\mathbf{u} = (u, v)$ is divergence free, we get

$$\left(u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y}\right)\delta x\,\delta y \times 1$$

The diffusion of C across the surface of the control volume is (by analogy with the development of surface forces that led to Eq. (2.65a))

$$\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)\cdot\left[-D\left(\frac{\partial C}{\partial x}\mathbf{i} + \frac{\partial C}{\partial y}\mathbf{j}\right)\right]\delta x\,\delta y \times 1$$

Summing the three contributions leads to the result given in part (a). The equation is

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right)$$

Part (b) asks about the necessity of assuming a dilute suspension of contaminant. First of all we assumed the flow field was unchanged. This is the key for the necessity of the assumption that the contaminant is dilute. In addition, the diffusion coefficient may be a function of the thermodynamic state. A first cut at dealing with this is to keep the derivatives of D in the formula. A second step would be to take into account the changes in density of the fluid particle to determine whether or no the changes in density are sufficient to induce natural convection. More extensive treatments of mass transport can be found in the literature; a good source is the book by Bird, Stewart and Lightfoot [4]. To take into account contaminant generation it is handled like a body force term in the momentum equation. Hence, the equation in (a) is altered as follows:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) + \dot{m}_c$$

Problem 2.4: In this problem we are interested in the components of the momentum equation for an inviscid pressure-driven flow subjected to a conservative body force. Le us assume that the control volume in this problem is the same as given in the solution of Problem 2.1. Also in the solution of Problem 2.1 the mass rate of flow through each face of the control surface that completely surrounds the control volume is given. Let us assume that there are two components of an external body force applied to the element of fluid and they are the compnenets of the vector $\mathbf{g} = (g_r, g_z)$. Let us also assume that the only surface force acting on the surface of the control volume is pressure. Thus, we have neglected viscous normal and shear stresses (i.e., we assume the flow is inviscid). There are two components of the momentum equation that are not zero because we assumed axisymmetric flow with zero swirl; as we have done in Problem 2.1. The radial and axial components of the momentum principle, respectively, are:

$$\rho\frac{\partial u}{\partial t}2\pi r dr dz + \dot{m}_R u_R - \dot{m}_L u_L + \dot{m}_T u_T - \dot{m}_B u_B = p_L A_L - p_R A_R + \rho g_r 2\pi r dr dz$$

$$\rho\frac{\partial w}{\partial t}2\pi r dr dz + \dot{m}_R w_R - \dot{m}_L w_L + \dot{m}_T w_T - \dot{m}_B w_B = p_B A_B - p_T A_T + \rho g_\phi 2\pi r dr dz$$

The velocity components in these formulas are

$$u_R = u + \frac{\partial u}{\partial z}\frac{dz}{2}, \quad u_L = u - \frac{\partial u}{\partial z}\frac{dz}{2}$$

$$u_T = u + \frac{\partial u}{\partial r}\frac{dr}{2}, \quad u_B = u - \frac{\partial u}{\partial r}\frac{dr}{2}$$

$$w_R = w + \frac{\partial w}{\partial z}\frac{dz}{2}, \quad w_L = w - \frac{\partial w}{\partial z}\frac{dz}{2}$$

$$w_T = w + \frac{\partial w}{\partial r}\frac{dr}{2}, \quad w_B = w - \frac{\partial w}{\partial r}\frac{dr}{2}$$

Substituting for the mass rate of flow and for the velocity components associated with each surface of the control volume into the momentum equations above, after rearranging terms and applying the continuity equation given in the problem statement of Problem 2.1, we get

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \rho g_r$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z$$

which are the equations of motion sought. These are the components of Euler's equation for axisymmetric flow.

Problem 2.5: Section 2.7 gives the solution procedure. The first set is to interpret u as the radial component of the velocity, and, hence, x as the radial coordinate, and interpret v as the z component of the velocity (w) and, hence, y as the axial coordinate. The main additions that need to be included are that area through the control surface perpendicular to the r direction increases with r and is equal to $2\pi r d\phi$, where ϕ is the coordinate in the angular direction. Axisymmetry implies that there are no changes in properties of the flow in the ϕ direction. Also assumed is that there is no angular velocity component. This does not mean that there is no rate of strain in the ϕ direction. In fact, for a fix δr and δz for $r_1 < r^2$ the control volume is larger for r_2 as compared with the control volume around r_1. Thus, transporting the flow radially leads to $\dot{\epsilon}_{\phi\phi} = u/r$. Otherwise, the derivation of the rates of strain follow the development given in Section 2.7.

In Part (b) of this problem is to transform the (x, y, z) form of the Navier-Stokes equations to cylindrical polar coordinates (r, ϕ, z) such that the changes in any property in ϕ are zero. Also, assume the velocity vector is $\mathbf{u} = (u, 0, w)$ in the polar coordinates. Since differentiation of the unit vector $\mathbf{k}$, which is in the z direction, is zero, the formula for the $\nabla \cdot \nabla f = \nabla^2 f$ term given in the solution of Problem 2.2 can be applied directly to get $\nabla^2 w$, the last term in the last equation given in the problem statement. If you take the second derivative of the first two terms in the first equation in the problem statement of Problem 2.2, you get the correct form for the third, fourth and fifth terms on the right hand side of the next to last equation in the problem statement for this problem.

Problem 2.6: Euler's equations for two-dimensional flows can be transformed from (x, y) to (r, ϕ) by applying the formulas given in the solution for Problem 2.2. Of course, the body force components must be converted from (g_x, g_y) to (g_r, g_ϕ) by similar formulas for u_r and u_ϕ given in the solution of Problem 2.2.

Problem 2.7: Care is required to draw the fluid particle and how it moves (as suggested in the hint). The vorticity can be found as given in one of the extensions to the solution of Problem 2.2. To examine the other rates of strain it may be convenient for the student to start with the particle in Fig. 2.13. An alternative approach is to apply the transformation equations in the solution of Problem 2.2 to the rates of strain in (x, y). The best treatment of the transformation relations as they apply to vectors and tensors is given in an appendix in Bird, Stewart and Lightfoot [4].

Problem 2.8: In this problem the shear stress is tangent to circles of radius r. The shear stress is equal to $\tau = \mu R\omega/h$. The area on which it acts is $2\pi r\, dr \times L$. Thus, the torque associated with this stress (force per unit area) is $dT = \tan 2\pi r^2\, dr \times L$. This needs to be integrated from $r = 0$ to $r = R$ to obtain the formula given. The power is equal to $2\pi nT = T\omega$. This is also what is given.

Problem 2.9: The key to this solution is to start, as indicated in the problem statement, with the formulas for the two compnenets of the Navier-Stokes equations in cylindrical coordinates given in Problem 2.5. You need to replace the formula for v to $v = -az/\zeta^2$. This is to take into account the fact that areas perpendicular to r are equal to $2\pi r$. Otherwise, the method is exactly what is presented in Section 2.10.3.

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1 Solutions to Chapter 2 problems

This section provides solutions for typical homework problems at the end of Chapter 2.

Problem 2.1: In this problem we are interested in the continuity equation for axisymmetric flow in terms of the cylindrical coordinate system (r, ϕ, z), where all flow variables are independent of angular coordinate, ϕ. Let the velocity components $(u, v, w) = (u, 0, w)$ in the (r, ϕ, z) coordinate directions. **(a)** Show that the continuity equation is given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

Solution: Let us consider an annular control volume with square cross-sectional area equal to $drdz$ centered at (r, z). Let us label the sides of the control volume R, L, T, B the centers of which are located at $(r, z + dz/2)$, $(r, z - dz/2)$, $(r + dr/2, z)$, $(r - dr/2, z)$, respectively. In this analysis the horizontal coordinate is z and the vertical (or radial) coordinate is r. The flow through each of the surfaces is

$$\dot{m}_R = \rho \left(w + \frac{\partial w}{\partial z}\frac{dz}{2} \right) 2\pi r\, dr$$

$$\dot{m}_L = \rho \left(w - \frac{\partial w}{\partial z}\frac{dz}{2} \right) 2\pi r\, dr$$

$$\dot{m}_T = \rho \left(u + \frac{\partial u}{\partial r}\frac{dr}{2} \right) 2\pi \left(r + \frac{dr}{2} \right) dz$$

$$\dot{m}_B = \rho \left(u - \frac{\partial u}{\partial r}\frac{dr}{2} \right) 2\pi \left(r - \frac{dr}{2} \right) dz$$

The conservation of mass principle means that the net flow of mass out of the control volume is zero, i.e.,

$$\dot{m}_R - \dot{m}_L + \dot{m}_T - \dot{m}_B = 0$$

This holds for both steady and unsteady conditions because ρ is assumed to be constant. An incompressible flow is volume preserving and, hence, this result is independent of whether or not the flow is steady. This is not the case for Euler's equation of motion; see Exercise 2.4. Substituting the expressions for the mass flow across each part of the control surface and rearranging terms, we get the equation given above.

(b) Show that the Stokes's stream function ψ defined by the following expressions for the velocity components

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

By direct substitution of these expressions into the continuity equation, we see that the continuity equation is automatically satisfied. Note that the function ψ is assumed to be a function for which the order of differentiation is immaterial.

Problem 2.2: In this problem we want to transform the continuity equation given in (x, y) to polar coordinates (r, ϕ). The two coordinate systems are related by the following formulas: $x = r\cos\phi$. $y = r\sin\phi$. Let the components of the velocity in (x, y) be given by $\mathbf{u} = (u_x, u_y)$ and the components in (r, ϕ) by $\mathbf{u} = (u, v)$. Note that

$$u_x = u\cos\phi - v\sin\phi, \quad u_y = u\sin\phi + v\cos\phi$$

We want to transform

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

to the following form:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} = 0$$

What we need to recall is that the derivative of any property f transfprms as follows:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$

From $x = r\cos\phi$ and $y = r\sin\phi$ we get

$$\frac{\partial x}{\partial x} = \frac{\partial r}{\partial x}\cos\phi - r\sin\phi\frac{\partial \phi}{\partial x}$$

$$\frac{\partial y}{\partial y} = \frac{\partial r}{\partial y}\sin\phi + r\cos\phi\frac{\partial \phi}{\partial y}$$

Since $r^2 = x^2 + y^2$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y$$

Hence,

$$\frac{\partial r}{\partial x} = \cos\phi, \quad \frac{\partial r}{\partial y} = \sin\phi$$

Thus,

$$\frac{\partial \phi}{\partial x} = -\frac{1-\cos^2\phi}{r\sin\phi} = -\frac{\sin\phi}{r}, \quad \frac{\partial \phi}{\partial y} = \frac{1-\sin^2\phi}{r\cos\phi} = \frac{\cos\phi}{r}$$

Applying these formulas, we get

$$\frac{\partial u_x}{\partial x} = \cos\phi\frac{\partial\left(u\cos\phi - v\sin\phi\right)}{\partial r} - \frac{\sin\phi}{r}\frac{\partial\left(u\cos\phi - v\sin\phi\right)}{\partial \phi}$$

$$\frac{\partial u_y}{\partial y} = \sin\phi\frac{\partial\left(u\sin\phi + v\cos\phi\right)}{\partial r} + \frac{\cos\phi}{r}\frac{\partial\left(u\sin\phi + v\cos\phi\right)}{\partial \phi}$$

Adding these equations and equating them to zero, we get the result sought.

Problem 2.4: In this problem we are interested in the components of the momentum equation for an inviscid pressure-driven flow subjected to a conservative body force. Le us assume that the control volume in this problem is the same as given in the solution of Problem 2.1. Also in the solution of Problem 2.1 the mass rate of flow through each face of the control surface that completely surrounds the control volume is given. Let us assume that there are two components of an external body force applied to the element of fluid and they are the compnenets of the vector $\mathbf{g} = (g_r, g_z)$. Let us also assume that the only surface force acting on the surface of the control volume is pressure. Thus, we have neglected viscous normal and shear stresses (i.e., we assume the flow is inviscid). There are two components of the momentum equation that are not zero because we assumed axisymmetric flow with zero swirl; as we have done in Problem 2.1. The radial and axial components of the momentum principle, respectively, are:

$$\rho\frac{\partial u}{\partial t}2\pi r dr dz + \dot{m}_R u_R - \dot{m}_L u_L + \dot{m}_T u_T - \dot{m}_B u_B = p_L A_L - p_R A_R + \rho g_r 2\pi r dr dz$$

$$\rho\frac{\partial w}{\partial t}2\pi r dr dz + \dot{m}_R w_R - \dot{m}_L w_L + \dot{m}_T w_T - \dot{m}_B w_B = p_B A_B - p_T A_T + \rho g_\phi 2\pi r dr dz$$

The velocity components in these formulas are

$$u_R = u + \frac{\partial u}{\partial z}\frac{dz}{2}, \quad u_L = u - \frac{\partial u}{\partial z}\frac{dz}{2}$$

$$u_T = u + \frac{\partial u}{\partial r}\frac{dr}{2}, \quad u_B = u - \frac{\partial u}{\partial r}\frac{dr}{2}$$

$$w_R = w + \frac{\partial w}{\partial z}\frac{dz}{2}, \quad w_L = w - \frac{\partial w}{\partial z}\frac{dz}{2}$$

$$w_T = w + \frac{\partial w}{\partial r}\frac{dr}{2}, \quad w_B = w - \frac{\partial w}{\partial r}\frac{dr}{2}$$

Substituting for the mass rate of flow and for the velocity components associated with each surface of the control volume into the momentum equations above, after rearranging terms and applying the continuity equation given in the problem statement of Problem 2.1, we get

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \rho g_r$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z$$

which are the equations of motion sought. These are the components of Euler's equation for axisymmetric flow.